

ON FREE ENERGIES OF THE POTTS MODEL ON THE CAYLEY TREE

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ABSTRACT. For the Potts model on the Cayley tree, some explicit formulae of the free energies and entropies (according to vector-valued boundary conditions (BCs)) are obtained. They include translation-invariant, periodic, Dobrushin-like BCs, as well as those corresponding to weakly periodic Gibbs measures.

Mathematics Subject Classifications (2010). 82B26 (primary); 60K35 (secondary)

Key words. Cayley tree, Potts model, boundary condition, Gibbs measure, free energy, entropy.

1. INTRODUCTION AND DEFINITIONS

On Cayley trees, not only Gibbs measures but also the free energy (and the entropy) depend on the BC. A study of this dependence is given in [1]. It is shown there that for all previously known BCs the free energies exist. Later, in [2] a construction of new Gibbs measures (called alternating Gibbs measures) is presented and their corresponding free energies are given. Moreover, it was proved that free energy of some alternating Gibbs measures may not exist.

The purpose of this paper is to study free energies of the Potts model on the Cayley tree.

The Cayley tree Γ^k (See [2]) of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles, from each vertex of which exactly $k + 1$ edges issue. Let $\Gamma^k = (V, L, i)$, where V is the set of vertices of Γ^k , L is the set of edges of Γ^k and i is the incidence function associating each edge $l \in L$ with its endpoints $x, y \in V$. If $i(l) = \{x, y\}$, then x and y are called *nearest neighboring vertices*, and we write $l = \langle x, y \rangle$.

The distance $d(x, y)$, $x, y \in V$ on the Cayley tree is defined by

$$d(x, y) = \min\{d \mid \exists x = x_0, x_1, \dots, x_{d-1}, x_d = y \in V \text{ such that } \langle x_0, x_1 \rangle, \dots, \langle x_{d-1}, x_d \rangle\}.$$

For the fixed $x^0 \in V$ we set $W_n = \{x \in V \mid d(x, x^0) = n\}$,

$$V_n = \{x \in V \mid d(x, x^0) \leq n\}, \quad L_n = \{l = \langle x, y \rangle \in L \mid x, y \in V_n\}.$$

It is known that there exists a one-to-one correspondence between the set V of vertices of the Cayley tree of order $k \geq 1$ and the group G_k of the free products of $k + 1$ cyclic groups $\{e, a_i\}$, $i = 1, \dots, k + 1$ of the second order (i.e. $a_i^2 = e$, $a_i^{-1} = a_i$) with generators a_1, a_2, \dots, a_{k+1} .

Denote by $S(x)$ the set of *direct successors* of $x \in G_k$. Let $S_1(x)$ be the set of all nearest neighboring vertices of $x \in G_k$, i.e. $S_1(x) = \{y \in G_k : \langle x, y \rangle\}$ and x_\downarrow denotes the unique element of the set $S_1(x) \setminus S(x)$.

We consider models where spin takes values from the set $\Phi = \{1, 2, \dots, q\}$, $q \geq 2$. A configuration σ is defined as a function $x \in V \rightarrow \sigma(x) \in \Phi$; the set of all configurations coincides with $\Omega = \Phi^V$.

The Hamiltonian of the Potts model has the form

$$H(\sigma) = -J \sum_{\langle x, y \rangle \in L} \delta_{\sigma(x)\sigma(y)}, \quad (1.1)$$

where $J \in \mathbb{R}$, δ_{uv} is the Kronecker symbol.

We identify the set Φ by the set $\{\sigma_1, \dots, \sigma_q\}$, where $\sigma_i \in \mathbb{R}^{q-1}$ such that

$$\sigma_i \sigma_j = \begin{cases} -\frac{1}{q-1}, & \text{if } i \neq j \\ 1, & \text{if } i = j. \end{cases}$$

Then we have

$$\delta_{\sigma(x)\sigma(y)} = \frac{q-1}{q} \left(\sigma(x)\sigma(y) + \frac{1}{q-1} \right). \quad (1.2)$$

Using this formula the Hamiltonian of the Potts model can be reduced to the Hamiltonian of the Ising model with q spin values:

$$H(\sigma) = -J \sum_{\langle x, y \rangle \in L} \sigma(x)\sigma(y). \quad (1.3)$$

We fix a basis $\{e_1, \dots, e_{q-1}\}$ on \mathbb{R}^{q-1} , such that $e_i = \sigma_i$, $i = 1, 2, \dots, q-1$. It is clear that

$$\sum_{i=1}^q \sigma_i = 0. \quad (1.4)$$

We note that if $h = (h_1, \dots, h_{q-1})$, then

$$h\sigma_i = \begin{cases} \frac{q}{q-1}h_i - \frac{1}{q-1} \sum_{j=1}^{q-1} h_j, & \text{if } i = 1, \dots, q-1, \\ -\frac{1}{q-1} \sum_{j=1}^{q-1} h_j, & \text{if } i = q. \end{cases}$$

Define a finite-dimensional distribution of a probability measure μ in the volume V_n as

$$\mu_n(\sigma_n) = Z_n^{-1} \exp \left\{ -\beta H_n(\sigma_n) + \sum_{x \in W_n} h_x \sigma(x) \right\}, \quad (1.5)$$

where $\beta = 1/T$, $T > 0$ is the temperature, $h_x \in \mathbb{R}^{q-1}$,

$$H_n(\sigma_n) = -J \sum_{\langle x, y \rangle \in L_n} \sigma(x)\sigma(y)$$

and Z_n^{-1} is the normalizing factor, i.e.

$$Z_n = Z_n(\beta, h) = \sum_{\sigma_n \in \Omega_n} \exp \left(-\beta H_n(\sigma_n) + \sum_{x \in W_n} h_x \sigma(x) \right).$$

The collection of vectors $h = \{h_x \in R^{q-1}, x \in V\}$ stands for (generalized) BC. The following limit (if it exist) is called *free energy* corresponding to BC h :

$$E(\beta, h) = - \lim_{n \rightarrow \infty} \frac{1}{\beta |V_n|} \ln Z_n(\beta, h).$$

We say that probability distributions (1.5) are compatible if for all $n \geq 1$ and $\sigma_{n-1} \in \Phi^{V_{n-1}}$ we have

$$\sum_{\sigma^{(n)} \in \Phi^{W_n}} \mu_n(\sigma_{n-1} \vee \sigma^{(n)}) = \mu_{n-1}(\sigma_{n-1}), \quad (1.6)$$

where $\sigma_{n-1} \vee \sigma^{(n)}$ is the concatenation of the configurations.

In this case, there exists a unique μ on Φ^V such, that for all n and $\sigma_n \in \Phi^{V_n}$ we have

$$\mu(\{\sigma|_{V_n} = \sigma_n\}) = \mu_n(\sigma_n).$$

Such measure is called a *limiting Gibbs measure* corresponding to Hamiltonian (1.3) and to the vector-valued function $h_x, x \in V$.

The next statement describes the condition on h_x ensuring that $\mu_n(\sigma_n)$ are compatible.

Theorem 1. *Measures (1.5) satisfy (1.6) if only if for all $x \in V \setminus \{x^0\}$ the following equation holds:*

$$h_x = \sum_{y \in S(x)} F(h_y, \theta), \quad (1.7)$$

where $F : h = (h_1, \dots, h_{q-1}) \in R^{q-1} \rightarrow F(h, \theta) = (F_1, \dots, F_{q-1}) \in R^{q-1}$ is defined as

$$F_i = \ln \left(\frac{(\theta - 1)e^{h_i} + \sum_{j=1}^{q-1} e^{h_j} + 1}{\theta + \sum_{j=1}^{q-1} e^{h_j}} \right), \quad \theta = \exp(J\beta).$$

Proof. In [8, p.106] this theorem is proved for Hamiltonian (1.1) (i.e. without change (1.2)). Here we shall prove (1.7) for Hamiltonian (1.3), i.e. with change (1.2). We show this because, a formula obtained in this proof will be helpful to find a general form of the free energy.

Substituting (1.5) in (1.6), in view of (1.2) we get

$$\begin{aligned} \frac{Z_{n-1}}{Z_n} \sum_{\sigma^{(n)}} \exp \left(J\beta \sum_{x \in W_{n-1}} \left(\sum_{y \in S(x)} \sigma(x)\sigma(y) \right) + \sum_{x \in W_{n-1}} \left(\sum_{y \in S(x)} h_y \sigma(y) \right) \right) = \\ \exp \left(\sum_{x \in W_{n-1}} h_x \sigma(x) \right), \sigma(x) \in \Phi. \end{aligned}$$

Consequently,

$$\frac{Z_{n-1}}{Z_n} \prod_{x \in W_{n-1}} \prod_{y \in S(x)} \sum_{\sigma(y)} \exp(J\beta\sigma(x)\sigma(y) + h_y\sigma(y)) = \prod_{x \in W_{n-1}} \exp(h_x\sigma(x)), \sigma(x) \in \Phi. \quad (1.8)$$

Fix $x \in W_{n-1}$ and rewrite (1.8) for $\sigma(x) = \sigma_i$, $i = 1, \dots, q-1$ and $\sigma(x) = \sigma_q$. Then dividing each of them by the last one we get

$$\prod_{y \in S(x)} \frac{\sum_{\sigma(y)} \exp((J\beta\sigma_i + h_y)\sigma(y))}{\sum_{\sigma(y)} \exp((J\beta\sigma_q + h_y)\sigma(y))} = \frac{\exp(h_x\sigma_i)}{\exp(h_x\sigma_q)}, i = 1, 2, \dots, q-1. \quad (1.9)$$

Notice that $h_x\sigma_i - h_x\sigma_q = \frac{q}{q-1}h_{x,i}$, where $h_x = (h_{x,1}, \dots, h_{x,q-1})$. Now we change the variables as $\frac{q}{q-1}h_{x,i} \rightarrow h_{x,i}$ then we get the equation (1.7). For a proof of the inverse statement see [8, p.107]. \square

Let $G_k/G_k^* = \{H_1, \dots, H_r\}$ be the quotient group, where G_k^* is a normal subgroup of index $r \geq 1$.

Definition 1. A set of vectors $h = \{h_x, x \in G_k\}$ is said to be G_k^* periodic, if $h_{yx} = h_x$ for any $x \in G_k$ and $y \in G_k^*$.

Definition 2. A set of vectors $h = \{h_x, x \in G_k\}$ is said to be G_k^* weakly periodic, if $h_x = h_{ij}$ for $x \in H_i$ and $x_{\downarrow} \in H_j$ for any $x \in G_k$.

Definition 3. A measure μ is said to be G_k^* -periodic (weakly periodic), if it corresponds to the G_k^* -periodic (weakly periodic) set of vectors h . The G_k -periodic measure is said to be translation-invariant.

In this paper we compute free energies which correspond to translation-invariant, periodic, weakly periodic and some non-periodic BCs (Gibbs measures).

2. FORMULA OF FREE ENERGY

The following theorem gives a general form of the free energy.

Theorem 2. For BCs satisfying (1.6), the free energy is given by the formula

$$E(\beta, h) = - \lim_{n \rightarrow \infty} \frac{1}{|V_n|} \sum_{x \in V_n} a(x), \quad (2.1)$$

where

$$a(x) = \frac{1}{q\beta} \sum_{i=1}^q \ln \left(\sum_{u=1}^q \exp\{(J\beta\sigma_i + h_x)\sigma_u\} \right). \quad (2.2)$$

Proof. In view of (1.9) we have

$$\prod_{y \in S(x)} \sum_{u=1}^q \exp(J\beta\sigma_i\sigma_u + \sigma_u h_y) = b(x) \exp(\sigma_i h_x), \quad i = 1, \dots, q.$$

Multiplying all such equalities and using (1.4) we obtain

$$b^q(x) = \prod_{i=1}^q \prod_{y \in S(x)} \sum_{u=1}^q \exp(J\beta\sigma_i\sigma_u + \sigma_u h_y) = \prod_{y \in S(x)} \prod_{i=1}^q \sum_{u=1}^q \exp(J\beta\sigma_i + h_y)\sigma_u,$$

hence

$$b(x) = \prod_{y \in S(x)} \prod_{i=1}^q \left(\sum_{u=1}^q \exp(J\beta\sigma_i + h_y)\sigma_u \right)^{1/q}.$$

Let $A_n = \prod_{x \in W_n} b(x)$. It is clear that $Z_n = A_{n-1}Z_{n-1}$. Consequently

$$Z_n = \prod_{x \in W_{n-1}} b(x)Z_{n-1} = \prod_{x \in W_{n-1}} b(x) \prod_{x \in W_{n-2}} b(x)Z_{n-2} = \dots = \prod_{x \in V_{n-1}} b(x),$$

and

$$\ln Z_n = \sum_{x \in V_{n-1}} \ln b(x).$$

Hence we obtain that

$$a(x) = \frac{1}{q\beta} \sum_{i=1}^q \ln \left(\sum_{u=1}^q \exp\{(J\beta\sigma_i + h_x)\sigma_u\} \right).$$

□

3. FREE ENERGIES CORRESPONDING TO SOME BCs

3.1. Translation-invariant case. Consider translation-invariant set of vectors h_x , i.e. $h_x = h = (h_1, h_2, \dots, h_{q-1}) \in R^{q-1}, \forall x \in G_k$. Then from (1.7) we obtain

$$h_i = k \ln \left(\frac{(\theta - 1)e^{h_i} + \sum_{j=1}^{q-1} e^{h_j} + 1}{\theta + \sum_{j=1}^{q-1} e^{h_j}} \right), \quad i = 1, \dots, q-1. \quad (3.1)$$

Denoting $z_i = \exp(h_i), i = 1, \dots, q-1$, we get from (3.1)

$$z_i = \left(\frac{(\theta - 1)z_i + \sum_{j=1}^{q-1} z_j + 1}{\theta + \sum_{j=1}^{q-1} z_j} \right)^k, \quad i = 1, 2, \dots, q-1. \quad (3.2)$$

For $k = 2$ we denote

$$x_1(m) = \frac{\theta - 1 - \sqrt{(\theta - 1)^2 - 4m(q - m)}}{2m}, \quad x_2(m) = \frac{\theta - 1 + \sqrt{(\theta - 1)^2 - 4m(q - m)}}{2m}, \quad (3.3)$$

where

$$\theta \geq \theta_m = 1 + 2\sqrt{m(q - m)}, \quad m = 1, \dots, q-1.$$

It is easy to see that

$$\theta_m = \theta_{q-m} \quad \text{and} \quad \theta_1 < \theta_2 < \dots < \theta_{\lfloor \frac{q}{2} \rfloor - 1} < \theta_{\lfloor \frac{q}{2} \rfloor} \leq q + 1.$$

Let $k = 2, J > 0$, then the following statements are known (see [6]).

1. If $\theta < \theta_1$, then the system of equations (3.1) has a unique solution $h_0 = (0, 0, \dots, 0)$;
2. If $\theta_m < \theta < \theta_{m+1}$ for some $m = 1, \dots, [\frac{q}{2}] - 1$, then the system of equations (3.1) has solutions

$$h_0 = (0, 0, \dots, 0), \quad h_{1i}(s), \quad h_{2i}(s), \quad i = 1, \dots, \binom{q-1}{s},$$

$$h'_{1i}(q-s), \quad h'_{2i}(q-s), \quad i = 1, \dots, \binom{q-1}{q-s}, \quad s = 1, 2, \dots, m,$$

where $h_{ji}(s)$, (resp. $h'_{ji}(q-s)$) $j = 1, 2$ is a vector with s (resp. $q-s$) coordinates equal to $2 \ln x_j(s)$ (resp. $2 \ln x_j(q-s)$) and remaining $q-s-1$ (resp. $s-1$) coordinates equal to 0. The number of such solutions is equal to

$$1 + 2 \sum_{s=1}^m \binom{q}{s};$$

3. If $\theta_{[\frac{q}{2}]} < \theta \neq q+1$, then there are $2^q - 1$ solutions to (3.1);
4. If $\theta = q+1$ then the number of solutions is as follows

$$\begin{cases} 2^{q-1}, & \text{if } q \text{ is odd} \\ 2^{q-1} - \binom{q-1}{q/2}, & \text{if } q \text{ is even;} \end{cases}$$

5. If $\theta = \theta_m$, $m = 1, \dots, [\frac{q}{2}]$, ($\theta_{[\frac{q}{2}]} \neq q+1$) then $h_{1i}(m) = h_{2i}(m)$. The number of solutions is equal to

$$1 + \binom{q}{m} + 2 \sum_{s=1}^{m-1} \binom{q}{s}.$$

Thus any solution of (3.1) has the form

$$h = (\underbrace{h_*, h_*, \dots, h_*}_m, 0, 0, \dots, 0), \quad m \geq 0 \quad (3.4)$$

up to a permutation of coordinates.

In this subsection we shall calculate free energies and entropy $S(\beta, h) = -\frac{dE(\beta, h)}{dT}$ for the set of translation-invariant vectors $h_x = h$, with h given by (3.4).

Case $m = 0$. In this case $h = h_0 = (0, 0, \dots, 0) \in R^{q-1}$. From (2.1) we have

$$\begin{aligned} E_{TI}(\beta, h_0) &= -a(x) = -\frac{1}{q\beta} \sum_{i=1}^q \ln \left(\sum_{u=1}^q \exp(J\beta\sigma_i\sigma_u) \right) = \\ &= -\frac{1}{q\beta} \sum_{i=1}^q \ln \left(\exp(J\beta) + (q-1) \exp\left(\frac{J\beta}{1-q}\right) \right) = \\ &= -\frac{1}{q\beta} q \ln \left(\exp(J\beta) + (q-1) \exp\left(\frac{J\beta}{1-q}\right) \right) = -J - \frac{1}{\beta} \ln \left(1 + (q-1) \exp\left(\frac{Jq\beta}{1-q}\right) \right). \end{aligned}$$

The corresponding entropy has the form

$$S_{TI}(\beta, h_0) = -\frac{dE_{TI}(\beta, h_0)}{dT} = \ln \left(1 + (q-1) \exp \left(\frac{Jq\beta}{1-q} \right) \right) + \frac{Jq\beta \exp \left(\frac{Jq\beta}{1-q} \right)}{1 + (q-1) \exp \left(\frac{Jq\beta}{1-q} \right)}.$$

Case $m \neq 0$. Using (3.4) we shall calculate the free energy:

$$\begin{aligned} E_{TI}(\beta, m, h_x) &= -\lim_{n \rightarrow \infty} \frac{1}{|V_n|} \sum_{x \in V_n} a(x) = -a(x) = -\frac{1}{q\beta} \sum_{i=1}^q \ln \left(\sum_{u=1}^q \exp \{ (J\beta\sigma_i + h_x)\sigma_u \} \right) = \\ &= -\frac{q-m}{q\beta} \ln \left(m \cdot e^{\left(-\frac{J\beta}{q-1} + \frac{q-m}{q-1} h_*\right)} + e^{\left(J\beta - \frac{m}{q-1} h_*\right)} + (q-m-1) \cdot e^{\left(-\frac{J\beta}{q-1} - \frac{m}{q-1} h_*\right)} \right) - \\ &= \frac{m}{q\beta} \ln \left((m-1) \cdot e^{\left(-\frac{J\beta}{q-1} + \frac{q-m}{q-1} h_*\right)} + e^{\left(J\beta + \frac{q-m}{q-1} h_*\right)} + (q-m) \cdot e^{\left(-\frac{J\beta}{q-1} - \frac{m}{q-1} h_*\right)} \right). \end{aligned} \quad (3.5)$$

Taking into account $h_* = 2 \ln x_j(m)$, $j = 1, 2$ (where the $x_j(m)$ are defined by (3.3)) we calculate the corresponding entropy:

$$\begin{aligned} S_{TI}(\beta, m, h_x) &= -\frac{dE_{TI}(\beta, m, h_x)}{dT} = -E_{TI}(\beta, m, h_x)\beta + \\ &= \frac{J\beta(q-m)}{q \left(m \cdot e^{\left(-\frac{J\beta}{q-1} + \frac{q-m}{q-1} h_*\right)} + e^{\left(J\beta - \frac{m}{q-1} h_*\right)} + (q-m-1) \cdot e^{\left(-\frac{J\beta}{q-1} - \frac{m}{q-1} h_*\right)} \right)} \times \\ &\quad \left[m(1 + (q-m)A) e^{\left(-\frac{J\beta}{q-1} + \frac{q-m}{q-1} h_*\right)} + (1-q-mA) e^{\left(J\beta - \frac{m}{q-1} h_*\right)} + \right. \\ &\quad \left. (q-m-1)(1-mA) e^{\left(-\frac{J\beta}{q-1} - \frac{m}{q-1} h_*\right)} \right] + \\ &= \frac{J\beta m}{q \left((m-1) \cdot e^{\left(-\frac{J\beta}{q-1} + \frac{q-m}{q-1} h_*\right)} + e^{\left(J\beta + \frac{q-m}{q-1} h_*\right)} + (q-m) \cdot e^{\left(-\frac{J\beta}{q-1} - \frac{m}{q-1} h_*\right)} \right)} \times \\ &\quad \left[(m-1)(1 + (q-m)A) e^{\left(-\frac{J\beta}{q-1} + \frac{q-m}{q-1} h_*\right)} + ((m-q)A - 1) e^{\left(J\beta - \frac{m}{q-1} h_*\right)} + \right. \\ &\quad \left. (q-m)(1-mA) e^{\left(-\frac{J\beta}{q-1} - \frac{m}{q-1} h_*\right)} \right], \end{aligned} \quad (3.6)$$

where $A = \frac{2e^{J\beta}}{(q-1)(e^{J\beta - 2me^{\frac{h_*}{2}}} - 1)}$.

Remark 1. We note that the entropy for $2 \ln x_1(m)$ and $2 \ln x_2(m)$ can be calculated by the formula (3.6) replacing h_* by $2 \ln x_1(m)$ and $2 \ln x_2(m)$ respectively.

Remark 2. Notice that under any permutations of the coordinates of the vector h_x the free energy and entropy do not change.

In Fig.1 graphs of the free energies (3.5) are shown.

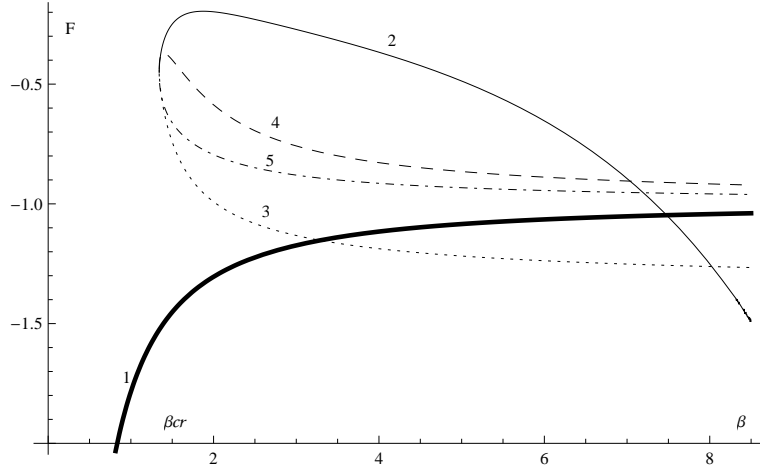


Fig. 1. Case $q = 3$. The free energy $F(\beta, h_0)$ the bold solid line (line 1); the free energy $F(\beta, (2 \ln x_1(1), 0))$ solid line (line 2); the free energy $F(\beta, (2 \ln x_2(1), 0))$ the dotted line (line 3); the free energy $F(\beta, (2 \ln x_1(2), 2 \ln x_1(2)))$ the dashed line (line 4); the free energy $F(\beta, (2 \ln x_2(2), 2 \ln x_2(2)))$ the dotted-dashed line (line 5), where $x_i(m), i = 1, 2, m = 1, 2$ defined in (3.3).

3.2. A non-translation-invariant BCs. In this subsection we consider non-translation-invariant BCs (Gibbs measures) constructed by N.Ganikhadjaev in [3]. Here one considers the half tree. Namely the root x^0 has k nearest neighbors. Consider an infinite path $\pi = \{x^0 = x_0 < x_1 < \dots\}$ (the notation $x < y$ meaning that paths from the root to y go through x). Take two different solutions h_*^1 and h_*^2 of (3.1) (having the form (3.4)). Associate to this path a collection h^π of vectors given by the condition

$$h_x^\pi = \begin{cases} h_*^1, & \text{if } x \prec x_n, x \in W_n, \\ h_*^2, & \text{if } x_n \preceq x, x \in W_n, \end{cases} \quad (3.7)$$

$n = 1, 2, \dots$ where $x \prec x_n$ (resp. $x_n \prec x$) means that x is on the left (resp. right) from the path π .

For a given infinite path π we put

$$\mathcal{W}_n^\pi = \{x \in W_n : x \prec x_n\},$$

where $n = 1, 2, \dots$.

Lemma 1. *For any π there exists the limit*

$$\lim_{n \rightarrow \infty} \frac{|\mathcal{W}_n^\pi|}{|W_n|} = a^\pi. \quad (3.8)$$

Proof. Put

$$\Delta_1^\pi = \mathcal{W}_1^\pi; \quad \Delta_n^\pi = \mathcal{W}_n^\pi \cap S(x_{n-1}),$$

where $n \geq 2$.

It is easy to see that

$$|\mathcal{W}_n^\pi| = \sum_{i=1}^n |\Delta_i^\pi| \cdot |W_{n-i}|$$

and $0 \leq |\Delta_i^\pi| \leq k-1$, for any i, π .

Let $a_n^\pi = \frac{|\mathcal{W}_n^\pi|}{|W_n|}$. Then using $|W_n| = k^{n-1}(k+1)$ we get

$$a_{n+1}^\pi - a_n^\pi = \frac{|\mathcal{W}_{n+1}^\pi|}{|W_{n+1}|} - \frac{|\mathcal{W}_n^\pi|}{|W_n|} = \frac{\Delta_{n+1}^\pi}{k^n(k+1)} \geq 0,$$

i.e. the sequence a_n^π is a monotone non-decreasing. It is easy to see that $a_n^\pi < 1$.

Consequently $\lim_{n \rightarrow \infty} \frac{|\mathcal{W}_n^\pi|}{|W_n|} = a^\pi$ exists and is finite. \square

Denote $V_n^l = \cup_{i=1}^n \mathcal{W}_i^\pi$. By the Stolz-Cesàro theorem (see e.g. [5]) we have

$$\lim_{n \rightarrow \infty} \frac{|V_n^l|}{|V_n|} = \lim_{n \rightarrow \infty} \frac{|\mathcal{W}_n^\pi|}{|W_n|} = a^\pi. \quad (3.9)$$

Now we calculate the free energy for the set of vectors h_x^π . In view of (3.9) we get

$$\begin{aligned} E_G(\beta, m, h_x^\pi) &= - \lim_{n \rightarrow \infty} \frac{1}{|V_n|} \sum_{x \in V_n} a(x) = - \frac{1}{q\beta} \lim_{n \rightarrow \infty} \frac{|V_n^l|}{|V_n|} \cdot \sum_{i=1}^q \ln \left(\sum_{u=1}^q \exp\{(J\beta\sigma_i + h_*^1)\sigma_u\} \right) - \\ &\quad - \frac{1}{q\beta} \lim_{n \rightarrow \infty} \left(1 - \frac{|V_n^l|}{|V_n|} \right) \cdot \sum_{i=1}^q \ln \left(\sum_{u=1}^q \exp\{(J\beta\sigma_i + h_*^2)\sigma_u\} \right) = \\ &\quad a^\pi E_{TI}(\beta, m, h_*^1) + (1 - a^\pi) E_{TI}(\beta, m, h_*^2), \end{aligned} \quad (3.10)$$

where $E_{TI}(\beta, m, h)$ is defined in (3.5).

Using (3.6) and (3.10) we get the following formula for the corresponding entropy:

$$S_G(\beta, m, h_x^\pi) = a^\pi S_{TI}(\beta, m, h_*^1) + (1 - a^\pi) S_{TI}(\beta, m, h_*^2).$$

3.3. Periodic BCs. In this subsection we consider periodic BCs and will be calculating free energies for them.

We consider the case $q = 3$. It is known (see [9]) that there are only $G_k^{(2)}$ -periodic Gibbs measures, where $G_k^{(2)}$ is the set of all words of even lengths. The corresponding set of vectors $h = \{h_x \in R^{q-1} : x \in G_k\}$ has the form

$$h_x = \begin{cases} \mathbf{h}^1, & \text{if } x \in G_k^{(2)}, \\ \mathbf{h}^2, & \text{if } x \in G_k \setminus G_k^{(2)}, \end{cases}$$

where $\mathbf{h}^1 = (h_1^1, h_2^1)$, $\mathbf{h}^2 = (h_1^2, h_2^2)$.

In view of (1.7) we have

$$\begin{cases} h_1^1 = k \ln \left(\frac{\theta \exp(h_1^2) + \exp(h_2^2) + 1}{\exp(h_1^2) + \exp(h_2^2) + \theta} \right) \\ h_2^1 = k \ln \left(\frac{\exp(h_1^2) + \theta \exp(h_2^2) + 1}{\exp(h_1^2) + \exp(h_2^2) + \theta} \right) \\ h_1^2 = k \ln \left(\frac{\theta \exp(h_1^1) + \exp(h_2^1) + 1}{\exp(h_1^1) + \exp(h_2^1) + \theta} \right) \\ h_2^2 = k \ln \left(\frac{\exp(h_1^1) + \theta \exp(h_2^1) + 1}{\exp(h_1^1) + \exp(h_2^1) + \theta} \right). \end{cases} \quad (3.11)$$

For $k = 3$, $J < 0$ it is known (see [4]) that if $0 < \theta < \frac{1}{4}$ then the system (3.11) has at least two solutions of the form: $h = (h^1, h^1, h^2, h^2)$. Now for such set of periodic vectors h_x , we shall describe free energies. From (2.2) we have

$$a(x) = \begin{cases} d(\mathbf{h}^1), & \text{if } x \in G_k^{(2)} \\ d(\mathbf{h}^2), & \text{if } x \in G_k \setminus G_k^{(2)}, \end{cases}$$

where

$$d(\mathbf{h}) = \frac{1}{q\beta} \sum_{i=1}^q \left(\sum_{u=1}^q \exp\{(J\beta\sigma_i + \mathbf{h})\sigma_u\} \right). \quad (3.12)$$

Denote

$$V_{even,n} = \{x \in V_n : x \in G_k^{(2)}\}, \quad V_{odd,n} = \{x \in V_n : x \in G_k \setminus G_k^{(2)}\}.$$

It is easy to check that for $n = 2p$,

$$|V_{even,2p}| = \frac{k^{2p+1} - 1}{k - 1}, \quad |V_{odd,2p}| = \frac{k^{2p} - 1}{k - 1},$$

for $n = 2p + 1$,

$$|V_{even,2p+1}| = \frac{k^{2p} - 1}{k - 1}, \quad |V_{odd,2p+1}| = \frac{k^{2p+2} - 1}{k - 1},$$

also we have

$$|V_n| = \frac{(k+1)k^n - 2}{(k-1)}.$$

Using these formulas we calculate the free energy:

$$\begin{aligned} E_{per}(\beta, h_x) &= - \lim_{n \rightarrow \infty} \frac{|V_{even,n}|d(\mathbf{h}^1) + |V_{odd,n}|d(\mathbf{h}^2)}{|V_n|} = \\ &= - \lim_{n \rightarrow \infty} \begin{cases} \frac{|V_{even,2p}|d(\mathbf{h}^1) + |V_{odd,2p}|d(\mathbf{h}^2)}{|V_{2p}|} & \text{if } n = 2p \\ \frac{|V_{even,2p+1}|d(\mathbf{h}^1) + |V_{odd,2p+1}|d(\mathbf{h}^2)}{|V_{2p+1}|} & \text{if } n = 2p + 1, \end{cases} \\ &= - \frac{1}{k+1} \begin{cases} kd(\mathbf{h}^1) + d(\mathbf{h}^2) & \text{if } n = 2p, p \rightarrow \infty \\ d(\mathbf{h}^1) + kd(\mathbf{h}^2) & \text{if } n = 2p + 1. \end{cases} \end{aligned}$$

From this equality it follows that if $d(\mathbf{h}^1) \neq d(\mathbf{h}^2)$, then for the periodic BCs a free energy does not exist. For $k = q = 3$ and fixed $\theta = \frac{1}{5}$, using a computer analysis one can see that $d(\mathbf{h}^1) \neq d(\mathbf{h}^2)$.

Remark 3. *It is known (see [1]) that for periodic Gibbs measures of the Ising model on Cayley trees the free energies exist. But for the Potts model we proved that free energy of periodic Gibbs measures may not exist.*

3.4. Weakly periodic BCs. Construct a weakly periodic BC. For $A \subset \{1, 2, \dots, k+1\}$ we consider $H_A = \{x \in G_k : \sum_{j \in A} w_j(x) \text{--even}\}$, where $w_j(x)$ is the number of a_j in a word x , $G_k/H_A = \{H_A, G_k \setminus H_A\}$ is a quotient group. For simplicity, we set $H_0 = H_A$, $H_1 = G_k \setminus H_A$. The H_A -weakly periodic sets of vectors $h = \{h_x \in R^{q-1} : x \in G_k\}$ have the following form

$$h_x = \begin{cases} h_1, & \text{if } x_\downarrow \in H_0, x \in H_0 \\ h_2, & \text{if } x_\downarrow \in H_0, x \in H_1 \\ h_3, & \text{if } x_\downarrow \in H_1, x \in H_0 \\ h_4, & \text{if } x_\downarrow \in H_1, x \in H_1. \end{cases} \quad (3.13)$$

Here $h_i = (h_{i1}, h_{i2}, \dots, h_{iq-1})$, $i = 1, 2, 3, 4$. By (1.7), we have

$$\begin{cases} h_1 = (k - |A|)F(h_1, \theta) + |A|F(h_2, \theta) \\ h_2 = (|A| - 1)F(h_3, \theta) + (k + 1 - |A|)F(h_4, \theta) \\ h_3 = (|A| - 1)F(h_2, \theta) + (k + 1 - |A|)F(h_1, \theta) \\ h_4 = (k - |A|)F(h_4, \theta) + |A|F(h_3, \theta). \end{cases} \quad (3.14)$$

In [7] it was shown that for $|A| = k$, $k \geq 6$ the system of equations (3.14) has at least two (not translation-invariant) solutions, which generate sets of vectors h_x of the form of (3.13), where all coordinates of vectors h_i , $i = 1, 2, 3, 4$ are equal and $h_i \neq h_j$ for $i \neq j$. Now for such weakly periodic sets of vectors h_x , we calculate the corresponding free energy.

We introduce the following

$$\begin{aligned} \mathcal{A}_n &= |\{\langle x, y \rangle \in L_n : x \in H_0, y = x_\downarrow \in H_0\}|, \\ \mathcal{B}_n &= |\{\langle x, y \rangle \in L_n : x \in H_0, y = x_\downarrow \in H_1\}|, \\ \mathcal{C}_n &= |\{\langle x, y \rangle \in L_n : x \in H_1, y = x_\downarrow \in H_0\}|, \\ \mathcal{D}_n &= |\{\langle x, y \rangle \in L_n : x \in H_1, y = x_\downarrow \in H_1\}|, \end{aligned}$$

where L_n is a set of edges in V_n .

It is known from [1] that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\mathcal{A}_n}{|V_n|} &= \lim_{n \rightarrow \infty} \frac{\mathcal{D}_n}{|V_n|} = \frac{1}{2(k+1)}, \\ \lim_{n \rightarrow \infty} \frac{\mathcal{B}_n}{|V_n|} &= \lim_{n \rightarrow \infty} \frac{\mathcal{C}_n}{|V_n|} = \frac{k}{2(k+1)}. \end{aligned}$$

Using these formulas we calculate the free energy:

$$E_{WP}(\beta, q, h_x) = - \lim_{n \rightarrow \infty} \frac{1}{|V_n|} (\mathcal{A}_n d(h_1) + \mathcal{B}_n d(h_2) + \mathcal{C}_n d(h_3) + \mathcal{D}_n d(h_4)) =$$

$$\frac{1}{2(k+1)} (E_{TI}(\beta, q-1, h_1) + kE_{TI}(\beta, q-1, h_2) + kE_{TI}(\beta, q-1, h_3) + E_{TI}(\beta, q-1, h_4)),$$

where $E_{TI}(\beta, m, h_x)$ is defined in (3.5). Now the entropy is

$$S_{WP}(\beta, q, h_x) = \frac{1}{2(k+1)} (S_{TI}(\beta, q-1, h_1) + kS_{TI}(\beta, q-1, h_2) +$$

$$kS_{TI}(\beta, q-1, h_3) + S_{TI}(\beta, q-1, h_4)),$$

where $S_{TI}(\beta, m, h_x)$ is defined in (3.6).

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